Calculating the self force in Schwarzschild spacetime by mode-sum regularization

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Abstract

We outline a method for calculating the self force (the "radiation reaction force") acting on a scalar particle in a strong field orbit in a Schwarzschild spacetime. In this method, the contribution to the self force associated with each multipole mode of the particle's field is calculated separately, and the sum over modes is then evaluated, subject to a certain regularization procedure. We present some explicit results concerning the implementation of the calculation scheme for a static particle, and also for a uniform circular motion, on the Schwarzschild background.

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This short manuscript deals with the calculation of the self force acting on a small charged object (a "particle") moving in a Schwarzschild spacetime. We shall briefly outline here a method, recently proposed by one of us (A.O.) [1], and present some results. This method involves the calculation of the contribution to the self-force due to each individual field's multipole, and the summation over these contributions, subject to a certain regularization procedure (outlined below). For simplicity we shall consider here a scalar charge. We expect, however, that the generalization to the electromagnetic self force will be almost straightforward. The generalization to the Kerr case and to the gravitational self force is likely to be possible as well. A closely related approach to the regularization of the self force, also based on the multipole expansion, is currently being studied by Lousto [2]. For an overview of further related works, the reader may refer, e.g., to Ref. [3].

The main challenge in calculating the self force is to deal with the various infinite quantities associated with the divergence of the field at the particle's location. Previously it has been proposed [4] that radiation reaction effects on the orbit may be calculated by evaluating the contribution of each Fourier-multipole mode ℓmw of the field to the radiative evolution, and then summing over all modes. This approach has two advantages: First, each individual mode of the field turns out to be continuous (and the corresponding self force to be bounded) even at the particle's location. Secondly, calculating each ℓmw mode of the field becomes a relatively simple task, as it only requires the solution of an ODE. In Ref. [4] this sum-over-modes approach has been proposed for the calculation of the adiabatic, orbit-integrated, evolution rate of the three constants of motion in Kerr: The Energy E, the azimuthal angular momentum L, and the Carter constant Q. One might have hoped that the same method could also be used for calculating the momentary self force. Unfortunately, at least for the self force, this naive procedure turns out, in general, inapplicable: Although each mode yields a finite contribution, the sum over all modes is found to diverge. This is the situation even in the simple case of a static particle in a flat space: For such a particle, located at a distance r_0 from the origin of coordinates (with respect to which the spherical harmonics are defined), the contribution of each multipole ℓ to the radial component of the self force is the same — $-q^2/(2r_0^2)$, where q is the particle's scalar charge. Obviously, the sum over ℓ diverges.[1] The regularization scheme outlined here is aimed to overcome this type of divergence (though in a nontrivial black-hole spacetime).

Practically, in this approach the calculation of the self force is composed of two separate parts:

(i) The numerical part: solving the appropriate ODE for each mode ℓmw of the field, and evaluating each mode's contribution to the self force[2] — a quantity we denote by $F^{\ell m\omega}$. (Alternatively, one may numerically solve the 1+1 PDE in the time domain, for each ℓ and m.)

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[1] Note, however, that the mode sum for the adiabatic, orbit-integrated, evolution rate of E and L does converge [4]. It is not clear yet whether the corresponding mode sum for Q converges or not.

[2] We use here $F_\mu = q\phi_{,\mu}$ as the basic expression for the force applied on a charged particle by the scalar field. Throughout this manuscript we shall often omit the vectorial index of the force and related quantitied for notational brevity.

(ii) The analytic part: applying a certain regularization procedure to the mode sum. This procedure involves the analytic calculation of certain regularization parameters, as we describe below.
This manuscript deals with the second part, namely, the analytic regularization scheme. The first part — the numerical determination of $F^{\ell m\omega}$ — is currently being implemented by Burko for several physical scenarios [5–8].

The analytic regularization scheme has already been fully implemented by one of us (A.O.) for circular orbits in the Schwarzschild geometry [10]. Barack and Ori [9] are currently developing the scheme for noncircular orbits (focusing on radial orbits at the moment). Since our regularization method has already been implemented in practical calculations (for static [5] and circular [6] orbits in the Schwarzschild geometry), we found it useful to provide a short account of the regularization scheme, and to briefly describe the main results already obtained. A more detailed account, including a systematic development of the method and a detailed calculation of the various parameters involved, will be given in Ref. [9].

As was shown by Quinn and Wald [3] (see also [11,12]), the physical self force is (in vacuum) the sum of two parts: (i) A local, Abraham-Lorentz-Dirac (ALD)-like term, and (ii) a "tail" term $F^{(\mathrm{tail})}$, associated with the tail part of the Green's function. The local term is trivial to calculate. We shall therefore focus here on the tail term. This term may be expressed as

$$F^{(\mathrm{tail})} \equiv \lim_{\epsilon\to 0^+} F_\epsilon, \tag{1}$$

where F_ϵ denotes the contribution to the force (evaluated at $\tau=0$) from the part $\tau \le -\epsilon$ of the particle's worldline. Decomposing F_ϵ into ℓ-modes, one finds

$$F^{(\mathrm{tail})} = \lim_{\epsilon\to 0^+} \sum_\ell F^\ell_\epsilon = \lim_{\epsilon\to 0^+} \sum_\ell \left(F^\ell - \delta F^\ell_\epsilon \right). \tag{2}$$

Here, F^ℓ_ϵ, δF^ℓ_ϵ, and F^ℓ denote the force associated with the ℓ-multipole of the field, sourced, respectively, by the interval $\tau \le -\epsilon$, the interval $\tau > -\epsilon$, and the entire world-line. The quantity F^ℓ may be identified with the sum over m, ω of the contributions from all modes ℓ, m, ω for a given ℓ. (Recall that in calculating a field's mode ℓ, m, ω one takes the source term to be the *entire* worldline. Since we are using the retarded Green's function, the part $\tau > 0$ does not contribute. However, the interval from $-\epsilon$ to 0^+ does contribute. Essentially, it is this part which is responsible to the instantaneous, divergent, piece of the Green's function, which should be removed from the expression for $F^{(\mathrm{tail})}$.)

A clarification is required here concerning the meaning of the last equality in Eq. (2): Let r_0 denote the value of r at the self force evaluation point. Then, the quantity F^ℓ_ϵ is well defined at $r=r_0$. The situation with F^ℓ and δF^ℓ_ϵ is more delicate, however. Each of these quantities has a well defined value at the limit $r \to r_0^-$, and a well defined value at the limit $r \to r_0^+$. Generically, for the r-component (and in some cases also for other components) these two one-sided values are not the same. Eq. (2) should thus be viewed as an equation for either the limit $r \to r_0^-$ of the relevant quantities (i.e. F^ℓ and δF^ℓ_ϵ), or the limit $r \to r_0^+$ of these quantities. Obviously, this equation is also valid for the *averaged force*, i.e. the average of these two one-sided values. In what follows we shall always consider the averaged force. [Of course, the final result of the calculation, $F^{(\mathrm{tail})}$ (which has a well defined value

at the evaluation point), should be the same regardless of whether it is derived from its one-sided limit $r \to r_0^-$, or from $r \to r_0^+$, or from their average.]

Next, we seek an ϵ-independent function h^ℓ, such that the series $\sum_\ell (F^\ell - h^\ell)$ converges. Once such a function is found, then Eq. (2) becomes

$$F^{(\mathrm{tail})} = \sum_\ell \left(F^\ell - h^\ell\right) - \lim_{\epsilon \to 0^+} \sum_\ell \left(\delta F^\ell_\epsilon - h^\ell\right). \tag{3}$$

In principle, h^ℓ can be found by investigating the asymptotic behavior of F^ℓ as $\ell \to \infty$. It is also possible, however, to derive h^ℓ from the large-ℓ asymptotic behavior of δF^ℓ_ϵ (the latter and F^ℓ must have the same large-ℓ asymptotic behavior, because their difference yields a convergent sum over ℓ). In addition to h^ℓ, the investigation of δF^ℓ_ϵ should also produce the parameter

$$d \equiv \lim_{\epsilon \to 0^+} \sum_\ell \left(\delta F^\ell_\epsilon - h^\ell\right), \tag{4}$$

which is required for the calculation of $F^{(\mathrm{tail})}$ in Eq. (3) .

From Eqs. (3) and (4) it is obvious that we only need the asymptotic behavior of δF^ℓ_ϵ in the immediate neighborhood of $\epsilon = 0$. It is therefore possible to analyze δF^ℓ_ϵ using local analytic methods. In particular, we can apply a perturbation analysis to the ℓ-mode field equation (in the time domain). To that end, we express the ℓ-mode effective potential $V^\ell(r)$ as a small perturbation $\delta V^\ell(r)$ over the value of $V^\ell(r)$ at the evaluation point, $V^\ell_0 \equiv V^\ell(r = r_0)$. This perturbative approach is used to analyze $G^\ell \equiv G^\ell[x^\alpha, x^\alpha_p(\tau)]$ (the ℓ-mode Green's function), where x^α is the evaluation point and $x^\alpha_p(\tau)$ describes the particle's worldline. Expressing G^ℓ as a function of τ and $z \equiv \tau\ell$, the perturbation analysis provides an expression for G^ℓ as a power series in τ (with z-dependent coefficients). Only the first three terms, i.e. the terms up to order τ^2, are required for the calculation of the self force (recall that eventually we take the limit $\epsilon \to 0$; as it turns out, in this limit the contributions from all higher-order terms vanish). The perturbation analysis yields explicit analytic expressions for the required three expansion coefficients (as functions of z), which we present in Ref. [9].

We next construct δF^ℓ_ϵ from G^ℓ (essentially by integrating the latter's gradient from $\tau = -\epsilon$ to $\tau = 0$). One then obtains the large-ℓ asymptotic form $\delta F^\ell_\epsilon = a\ell + b + c\ell^{-1} + O(\ell^{-2})$, in which the parameters a, b, and c are independent of ℓ and ϵ. (These parameters do depend on the evaluation point, and also on the orbit – through the values of the particle's four-velocity and its first few proper-time derivatives there.) The regularization function h^ℓ thus takes the form

$$h^\ell = a\ell + b + c\ell^{-1}. \tag{5}$$

Consequently, the tail part of the self force is found to be

$$F^{(\mathrm{tail})} = \sum_\ell \left(F^\ell - a\ell - b - c\ell^{-1}\right) - d. \tag{6}$$

The implementation of this regularization scheme thus amounts to analytically calculating the four regularization parameters a, b, c, and d, using the above mentioned perturbation expansion for the Green's function. (Note that there exist four such parameters for each vectorial component of the force.)

We shall now present the main results already obtained for the values of the above regularization parameters in various cases. These results concern the r-component of the self force, and are valid for either geodesic or non-geodesic orbits. It can be shown that the parameter a vanishes for all orbits in Schwarzschild [10], and preliminary results indicate that the same holds for c. (Recall that we consider here the averaged force: the "one-sided" values of a are, in general, nonzero [9].) On the other hand, the parameters b and d are generically nonvanishing. In what follows we give the values of these parameters for some physical scenarios.[3]

For static or circular orbits the parameter d (as well as a and c) is found to vanish [10,9]:

$$a_r = c_r = d_r = 0 \quad \text{for static or circular orbits,} \tag{7}$$

where the sub-index r indicates that these parameters are associated with the covariant r-component of the self force. Hence, in these cases one finds[4]

$$F_r^{(\mathrm{tail})} = \sum_\ell \left(F_r^\ell - b_r\right) \quad \text{(static or circular orbits).} \tag{8}$$

For a static particle, the parameter b_r is given by [9]

$$b_r = -\frac{q^2}{2r_0^2}\left(\frac{r_0 - M}{r_0 - 2M}\right) \quad \text{(static particle),} \tag{9}$$

where M is the black hole's mass. For a circular orbit, b_r takes the form [10]

$$b_r = -\frac{q^2}{2r_0^2}\,\frac{1}{u^t\sqrt{g_{tt}}}\,\left(2\,I_a - \frac{r_0 - 3M}{r_0 - 2M}\,I_b\right) \quad \text{(circular orbit),} \tag{10}$$

where

$$I_a = F(1/2, 1/2; 1; v^2) \quad , \quad I_b = F(1/2, 3/2; 1; v^2), \tag{11}$$

F denotes the hypergeometric function, and v is the tangential velocity with respect to a static observer: $v = \frac{d\varphi}{dt}\,\frac{r_0}{\sqrt{g_{tt}}}$ (for an equatorial orbit).

Recently, Burko numerically calculated $F^{\ell m\omega}$ for static [5] and circular [6] orbits, and used the above regularization scheme to calculate the self force. The above expressions (9,10) for b_r (as well as the vanishing of a_r and c_r) are in excellent agreement with the large-ℓ limit of his numerically-deduced F^ℓ (obtained by summing $F^{\ell m\omega}$ over m and ω).

For radial motion in flat space (i.e., motion with fixed angular coordinates θ and φ), the parameter b_r is found to be [9]

$$b_r = -\frac{q^2}{2r_0^2}\left(1 - \dot{r}^2 + r_0\ddot{r}\right) \quad \text{(radial motion in Minkowski),} \tag{12}$$

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[3] We use metric signature $(-+++)$ throughout.

[4] For static or circular orbits, the r-component of the local, ALD-like, term of the total self force vanishes. Thus, Eq. (8) describes, if fact, the (r-component of the) *total* self force.

where a dot denotes differentiation with respect to proper time. In this case, the parameter d does not vanish, in general, and one finds [9]

$$d_r = \frac{1}{3}q^2\left(\dot{a}_r - u_r a^2\right) \quad \text{(radial motion in Minkowski)}, \tag{13}$$

where u_α and a_α are the four-velocity and four-acceleration, respectively, and $a^2 \equiv a^\alpha a_\alpha$. Remarkably, d_r thus equals the r-component of the ALD-like local term (given in Ref. [3]). As a consequence, this contribution from the tail term will *cancel out* the r-component of the local ALD-like term in the expression for the total self force.

Generalization of the above results, Eqs. (12) and (13), to general radial motion in a Schwarzschild spacetime (and also in a more general class of static spherically symmetric spacetimes) will be presented in a forthcoming paper [9].

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